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DESIGN OF PLATES WITH GIVEN RIGIDITIES FROM A MINIMUM NUMBER OF LAYERS AND MATERIALS

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The problem of designing a plate with a given set of rigidities using a minimum number of layers, which, as it has been established, is equal to four, has been considered. We consider the cases where flexural rigidities, rigidities in the plate plane, and asymmetric rigidities and where only longitudinal and flexural rigidities, which are the physical characteristics of the plane, are given. It has been proved that in both cases the ranges of longitudinal and flexural rigidities are equal (thus the question of the role of asymmetric rigidities is solved).

The problem of designing laminated plates with a given set of rigidities has been considered by many authors, e.g., in [1–4]. In most works it was a problem of optimization of rigidity, weight, etc. The present paper considers the problem of designing a plate with a given set of rigidities. Primary consideration is given to the determination of the minimum number of materials and layers. Earlier, a similar problem was solved for the cases where the Young modulus of layers took on any values from the interval $[E_*, E^*]$ [4] or where the plate was formed from N layers of equal thickness [5]. The former is mainly of theoretical interest, since in reality it is impossible to dispose of an infinite set of materials. The second problem is quite practical; with a number of layers $N \sim 10$ it can be solved on a computer in a reasonable time (in a few minutes) [5]; however, at $N \sim 100$ it becomes essentially unsolvable.

Let us consider a laminated plate (Fig. 1). At coinciding Poisson coefficients ν of the materials of the layers (which takes place in many cases) the rigidities of the plate D^{μ}_{ijkl} (i, j, k, l = 1, 2) are expressed in terms of integrals (function $E(y)$ moments):

$$I_{\mu}(E) = \int_{-1/2}^{1/2} y^{\mu} E(y)\, dy\,, \quad \mu = 0, 1, 2\,.$$

The function $E(y)$ has the meaning of the Young modulus of the material, y being the coordinate across the plate. The plate rigidities are expressed in terms of intervals $I_{\mu}(E)$ (μ = 0, 1, 2) by the formulas [6]

$$D^{\mu}_{1111} = D^{\mu}_{2222} = \frac{h^{\mu+1}}{1-\nu^2} I_{\mu}(E)\,,$$

$$D^{\mu}_{1122} = D^{\mu}_{2211} = \frac{\nu h^{\mu+1}}{1-\nu^2} I_{\mu}(E)\,, \tag{1}$$

$$D^{\mu}_{1212} = D^{\mu}_{2121} = \frac{h^{\mu+1}}{1+\nu} I_{\mu}(E)\,.$$

Here D^{0}_{ijkl} denotes the rigidities in the plane of the plate; D^{1}_{ijkl} denotes asymmetric rigidities, and D^{2}_{ijkl} — flexural rigidities. Thus, the solution of the problem of designing a plate with given rigidities is equivalent to the solution of the problem on assigning given values a, b, c to three intervals (moments $I_{\mu}(E)$ (μ = 0, 1, 2) of the function $E(y)$.

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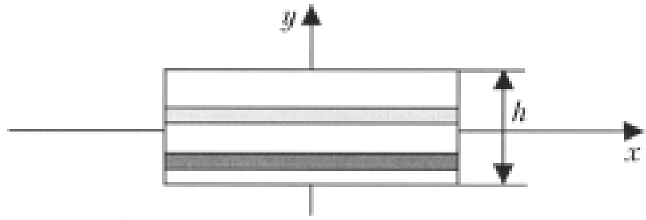

Fig. 1. Diagram of the laminated plate.

In [4], the solution of the design problem was obtained on a set of functions $U_E = \{(y)\}$, where $E(y)$ takes on any values from the interval $[E_*, E^*]$. This means that materials with any Young moduli from the interval $[E_*, E^*]$ are available to the designer. It was found that the solution of the problem

$$\int_{-1/2}^{1/2} E(y)\,dy = a\,, \tag{2}$$

$$\int_{-1/2}^{1/2} yE(y)\,dy \to \min_1/\max_1$$

has the form of a function formed from two steps (the first solution corresponds to the minimization problem, the second one — to the maximization problem):

$$E(y) = \begin{cases} E^*, & -1/2 \le y \le \lambda\,; \\ E_*, & \lambda \le y \le 1/2\,; \end{cases} \quad E(y) = \begin{cases} E_*, & -1/2 \le y \le \lambda\,; \\ E^*, & \lambda \le y \le 1/2\,, \end{cases} \tag{3}$$

where $-1/2 \le \lambda \le 1/2$. The solution of the problem

$$\int_{-1/2}^{1/2} E(y)\,dy = a\,, \quad \int_{-1/2}^{1/2} yE(y)\,dy = b\,, \tag{4}$$

$$\int_{-1/2}^{1/2} y^2 E(y)\,dy \to \min_2/\max_2$$

has the form of a step function (the first solution corresponds to the minimization problem, the second one — to the maximization problem):

$$E(y) = \begin{cases} E_*, & -1/2 \le y \le \lambda_1\,; \\ E^*, & \lambda_1 \le y < \lambda_2\,; \\ E_*, & \lambda_2 \le y \le 1/2\,; \end{cases} \quad E(y) = \begin{cases} E^*, & -1/2 \le y < \lambda_1\,; \\ E_*, & \lambda_1 \le y < \lambda_2\,, \\ E^*, & \lambda_2 \le y \le 1/2\,, \end{cases} \tag{5}$$

where $-1/2 \le \lambda_1 \le \lambda_2 \le 1/2$. We shall use further problems (2) and (4) and their solutions (3) and (5).

The designer practically has at his disposal a finite (discrete) set of materials for creating a plate. Therefore, in practical problems $E(y)$ assumes a discrete set of values $E_* = E_1, E_2, ..., E_n = E^*$, and the problem should be considered on the set of step functions

$$D = \left\{E(y): \; E(y) \cup \left\{E_* = E_1, E_2, ..., E_n = E^*\right\}\right\}. \tag{6}$$

To solve it, one has to know:

a) what values the integrals $I_\mu(E)$ ($\mu = 0, 1, 2$) can take on set (6);

b) how to plot the function assigning to the integrals $I_\mu(E)$ the given values of a, b, c (provided that a, b, c belong to the set of possible values of the integrals).

Further we shall need a set of step functions assuming only two values from the interval $[E_*, E^*]$ — the largest and the least ones:

$$D_*^* = \left\{E(y): \; E(y) \cup \left\{E_*, E^*\right\}\right\}.$$

Apparently, $D_*^* \subset D \subset U_E$.

Possible Values of Plate Rigidities. Let us determine the possible values of integrals $I_\mu(E)$ ($\mu = 0, 1, 2$) on the sets U_E, D, and D_*^*. The reference point for analysis is the fact that solutions (3) and (5) belong to D_*^*. This fact, which was not used before, enables one to make great progress in solving the problem of designing laminated plates.

Apparently, the largest and the least values of the integral $I_0(E)$ on the sets U_E, D, and D_*^* coincide and are equal to E_* and E^*. Let us show that on the set D_*^* the integral $I_0(E)$ takes all intermediate values between E_* and E^*. Let us consider a step function of the form (3) belonging to D_*^*. As the parameter λ is changed from $-\frac{1}{2}$ to $\frac{1}{2}$, the value of the integral $I_0(E)$ varies continuously from E_* to E^*. Then the solution of the problem

$$I_0(E) = a \; (E_* \le a \le E^*), \tag{7}$$

$$I_1(E) \to \max{}_1 / \min{}_1, \tag{8}$$

$$E \cup U_E \tag{9}$$

is function (3) belonging to the set D_*^*, i.e., the maximum and the minimum of $I_1(E)$ on U_E, D, and D_*^* are the same. Let us show that $I_1(E)$ takes all intermediate values between $\max_1$ and $\min_1$ for the functions from D_*^* on condition (7). In [4], the minimum and maximum values for the integral $I_1(E)$ on condition (7) were calculated for $E \cup U_E$:

$$\min{}_1 = -\frac{(E^* - a)(a - E_*)}{2(E^* - E_*)}, \quad \max{}_1 = \frac{(E^* - a)(a - E_*)}{2(E^* - E_*)}.$$

It is impossible to plot the function from D_*^* on which the integral takes all intermediate values with the use of the technique from [4]. Let us use another method. Let us introduce a function continuously depending on the parameter and going, upon its change, from the first function (3) into the second one. This transformation should not withdraw the functions from the set D_*^* (i.e., the possible values assumed by the function, that is, E_* and E^*) and for all values of the parameter condition (7) should be fulfilled.

Consider the following function $E(y)$ continuously depending on the parameter λ:

$$E(y) = \begin{cases} E_*, & -1/2 \le y < \lambda; \\ E^*, & \lambda \le y < \lambda + \dfrac{a - E_*}{E^* - E_*}; \\ E_*, & \lambda + \dfrac{a - E_*}{E^* - E_*} \le y \le 1/2. \end{cases} \tag{10}$$

At $\lambda = -\frac{1}{2}$, function (10) coincides with the first function in (3), and at $\lambda = \frac{1}{2} - \frac{a - E_*}{E^* - E_*}$ it coincides with the second function. The integral $I_0(E)$ of function (10) is equal to a for all $\lambda \cup \left[-\frac{1}{2}, \frac{1}{2} - \frac{a - E_*}{E^* - E_*}\right]$. The integral $I_1(E)$ of (10) is a continuous function of the variable λ and varies from $\min_1$ to $\max_1$.

Consider the problem

$$I_0 (E) = a \,, \tag{11}$$

$$I_1 (E) = b \,, \tag{12}$$

$$I_2 (E) \to \max_2 / \min_2 \,, \tag{13}$$

$$E \cup U \,, \tag{14}$$

whose solution has the form of function (5) belonging to the set D_*^*, i.e., the maximum and the minimum in problem (11)–(14) on U_E, D, and D_*^* are the same.

Because of the misprint in [2, 4] in the formulas for $\max_2$ and $\min_2$, let us make calculations of the maximum and minimum values of the integral $I_2(E)$. The minimum of $I_2(E)$ is attained on the first function from (5). In so doing, λ_1 and λ_2 should be chosen so that conditions (11) and (12) are fulfilled. For the first function from (5), equalities (11) and (12) take on the form

$$E_* + (\lambda_2 - \lambda_1)\,(E^* - E_*) = a \,, \tag{15}$$

$$(E^* - E_*)\,\frac{\lambda_2^2 - \lambda_1^2}{2} = b \,. \tag{16}$$

Solving system (15), (16), we find

$$\lambda_1 = \frac{b}{a - E_*} - \frac{a - E_*}{2\,(E^* - E_*)} \,, \quad \lambda_2 = \frac{b}{a - E_*} + \frac{a - E_*}{2\,(E^* - E_*)} \,.$$

Then

$$I_2 (E) = \min_2 = \frac{E_*}{12} + \frac{b^2}{a - E_*} + \frac{(a - E_*)^3}{12\,(E^* - E_*)^2} \,.$$

The maximum of $I_2(E)$ is attained on the second function from (5), and λ_1 and λ_2 thereby should be chosen so that conditions (11) and (12) are fulfilled for all λ_1 and λ_2. The conditions above for the second function from (5) take on the form

$$E_* + (E^* - E_*)\,(\lambda_1 + 1 - \lambda_2) = a \,, \tag{17}$$

$$(E^* - E_*)\,\frac{\lambda_1^2 - \lambda_2^2}{2} = b \,. \tag{18}$$

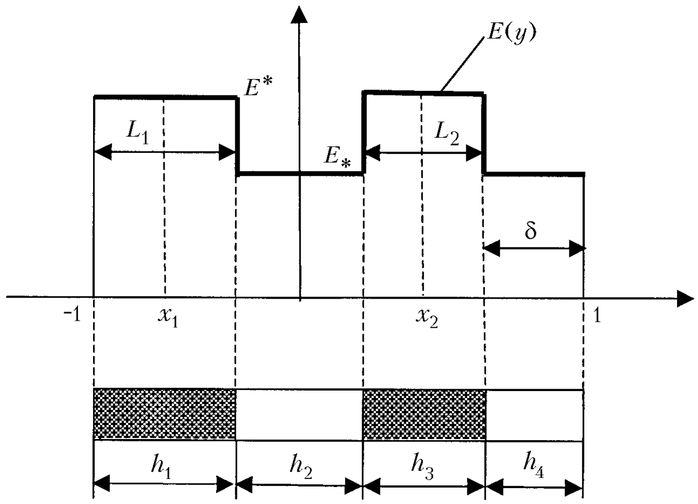

Fig. 2. Step function $E(y)$ and its corresponding design of the laminated plate (plate-thickness distribution of two materials).

The solution of system (17), (18) yields

$$\lambda_1 = \frac{b}{a - E^*} + \frac{a - E^*}{2\,(E^* - E_*)}\,, \quad \lambda_2 = \frac{b}{a - E^*} - \frac{a - E^*}{2\,(E^* - E_*)}\,,$$

$$I_2\,(E) = \max\nolimits_2 = \frac{E^*}{12} - \frac{b^2}{E^* - a} - \frac{(E^* - a)^3}{12\,(E^* - E_*)^2}\,.$$

Let us show that $I_2(E)$ takes all intermediate values between $\max_2$ and $\min_2$ for the functions from the set D_*^*. It is also impossible to use for this the technique from [4], and we shall again use the method described above. We shall indicate the transformation which is continuously dependent on the parameters and transfers the first function from (5) into the second one and at the same time does not withdraw the function from the set D_*^* and ensures the fulfillment of equalities (11), (12) for all parameter values.

Consider the step function

$$E\,(y) = \begin{cases} E_*\,, & -1/2 \le y < \lambda_1\,; \\ E^*\,, & \lambda_1 \le y < \lambda_2\,; \\ E_*\,, & \lambda_2 \le y < \lambda_3\,; \\ E^*\,, & \lambda_3 \le y < \lambda_4\,; \\ E_*\,, & \lambda_4 \le y \le 1/2\,, \end{cases} \tag{19}$$

where $-\frac{1}{2} < \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda < \frac{1}{2}$.

Let us see that at a proper choice of parameters λ_1, λ_2, λ_3, and λ_4 the transform of (19) has the required properties. For function (19), equalities (11) and (12) are written as

$$(\lambda_2 - \lambda_1)\,(E^* - E_*) + (\lambda_4 - \lambda_3)\,(E^* - E_*) + E_* = a\,, \tag{20}$$

$$(E^* - E_*)\,(\lambda_2 - \lambda_1)\,\frac{\lambda_2 + \lambda_1}{2} + (E^* - E_*)\,(\lambda_4 - \lambda_3)\,\frac{\lambda_4 + \lambda_3}{2} = b\,. \tag{21}$$

It is convenient to write equalities (20), (21) in variables $L_1 = \lambda_2 - \lambda_1$, $L_2 = \lambda_4 - \lambda_3$ having the meaning of the width of "high" steps (steps where the function takes the value of E^* are called "high" steps (Fig. 2)) and variables

$$x_1 = \frac{\lambda_2 + \lambda_1}{2}, \quad x_2 = \frac{\lambda_4 + \lambda_3}{2}, \tag{22}$$

corresponding to the coordinates of the middle of the steps. In these variables, Eqs. (2) and (21) have the form

$$(E^* - E_*)\,(L_1 + L_2) + E_* = a\,, \tag{23}$$

$$(E^* - E_*)\,(L_1 x_1 + L_2 x_2) = b\,. \tag{24}$$

Hence, for L_1 and L_2 we obtain

$$L_1 = \frac{b - x_2\,(a - E_*)}{(E^* - E_*)\,(x_1 - x_2)}, \quad L_2 = \frac{x_1\,(a - E_*) - b}{(E^* - E_*)\,(x_1 - x_2)}. \tag{25}$$

The condition $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ is equivalent to the set of conditions

$$x_2 - x_1 \ge \frac{L_1 + L_2}{2}, \quad L_1 > 0\,, \quad L_2 > 0\,, \tag{26}$$

which are fulfilled if the middles of the "high" steps x_1 and x_2 of (22) are no less than half of their length ($L_1/2$ and $L_2/2$) away from the middle of the "high" step x_{m} defined by the first formula of (5). The second function in (5) contains two "high" steps. Let us take their middles x_{10} and x_{20} as the initial values of x_1 and x_2, respectively. As x_1 is varied from x_{10} to $x_{\text{m}} - \frac{L_1}{2}$ and x_2 from x_{10} to $x_{\text{m}} + \frac{L_2}{2}$, the second function from (5) continuously transforms to the first one. In so doing, they will leave the set D_*^* and conditions (11) and (12) will be met, and the integral $I_2(E)$ will take on all intermediate values between $\max_1$ and $\min_1$. In the case under consideration, the transform depends on two variables x_1 and x_2, which we will subsequently take into account.

Way of Finding a Design of a Plate with Given Rigidities. To design a plate with given characteristics, one has to solve the problem

$$I_0\,(E) = a\,, \quad I_1\,(E) = b\,, \quad I_2\,(E) = c \tag{27}$$

for the integrand $E(y)$.

Above, the conditions of solvability of (27) have been established: the quantities a, b, c can take on the following values:

$$E_* \le a \le E^*\,,$$

$$-\frac{(E^* - a)\,(a - E_*)}{2\,(E^* - E_*)} \le b \le \frac{(E^* - a)\,(a - E_*)}{2\,(E^* - E_*)}, \tag{28}$$

$$\frac{E_*}{12} + \frac{b^2}{a - E_*} + \frac{(a - E_*)^3}{12\,(E^* - E_*)^2} \le c \le \frac{E^*}{12} - \frac{b^2}{E^* - a} - \frac{(E^* - a)^3}{12\,(E^* - E_*)^2}.$$

The above computations make it possible not only to obtain the possible values of rigidities but also to find a design of a plate with given rigidities provided that the rigidities satisfy conditions (28).

There exist an infinitely large number of solutions to the design problem. We are interested in the simplest solutions. In estimating the integral $I_2(E)$, we formed function (19) with five steps. And $I_2(E)$ thereby is a function of two variables, x_1 and x_2. So, in using (25) the first two equations in (27) are directly satisfied, and the third equation takes the form of the algebraic equation $I_2(x_1, x_2) = c$, where $I_2(x_1, x_2)$ is an analytically given function (i.e., it is given by a known formula). This algebraic equation can be solved numerically, which will permit obtaining a design of the plate.

Simplest Design of a Plate with Given Rigidities. By the simplest design is meant a design containing the minimum number of layers. Let us show that in the proposed method for obtaining designs from five layers, the number of layers can be reduced to four. To this end, let one variable in the function $I_2(x_1, x_2)$ remain free and choose the other so as to reduce the number of steps of the function $E(y)$ (the number of layers). Let us seek the solution in the form

$$E(y) = \begin{cases} E^*, & -1/2 \le y < A, \\ E_*, & A \le y < B, \\ E^*, & B \le y < 1/2 - \delta, \\ E_*, & 1/2 - \delta \le y \le 1/2, \end{cases} \tag{29}$$

where δ is a variable. In this case, the leftmost step continuously adjoins the left end of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (Fig. 2). The condition that the left "high" step at all δ adjoins the edge of $y = -\frac{1}{2}$ is written as follows: $x_1 = -\frac{1}{2} + \frac{L_1}{2}$, where L_1 is determined by formula (22).

The integrals $I_0(E)$ and $I_1(E)$ for function (29) can be written in a form analogous to (23), (24):

$$I_0(E) = E_* + (E^* - E_*)(L_1 + L_2),$$

$$I_1(E) = (E^* - E_*)(x_1 L_1 + x_2 L_2).$$

Then the equations $I_0(E) = a$ and $I_1(E) = b$ from (27) will take on the form

$$L_1 + L_2 = \frac{a - E_*}{E^* - E_*}, \quad x_1 L_1 + x_2 L_2 = \frac{b}{E^* - E_*}.$$

Here L_1 and L_2 are the widths and x_1 and x_2 are the "middles" of the high steps (Fig. 2).

Let us express the quantities x_1, x_2, L_1, and L_2 in terms of the new variable δ:

$$\begin{gathered} x_1 = x_1(\delta) = \frac{L_1(\delta)}{2} - \frac{1}{2}, \\ x_2 = x_2(\delta) = \frac{1}{2} - \delta^2 - \frac{L_2(\delta)}{2}, \\ L_1 = L_1(\delta) = \frac{\dfrac{b}{E^* - E_*} - \dfrac{M}{2} + \dfrac{M^2}{2} + M\delta}{M + \delta - 1}, \\ L_2 = L_2(\delta) = \frac{a - E_*}{E^* - E_*} - L_1(\delta), \end{gathered} \tag{30}$$

TABLE 1. Designs of Plates in Terms of Quantities L_1, L_2, x_1, and x_2

Pair of materials	L_1	L_2	x_1	x_2
(1, 9)	0.378	0.407	–0.311	0.289
(6, 10)	0.160	0.160	–0.420	0.420

TABLE 2. Designs of Plates in Terms of Quantities h_1, h_2, h_3, and h_4

Pair of materials	h_1	h_2	h_3	h_4
(1, 9)	0.377	0.208	0.407	0.008
(6, 10)	0.160	0.860	0.160	0

where $M = \dfrac{a - E_*}{E^* - E_*}$.

The integral $I_2(E)$ is expressed in terms of the above quantities as follows:

$$I_2(\delta) = \frac{E_*}{12} + (E^* - E_*)\, L_1(\delta) \left[x_1^2(\delta) + \frac{L_1^2(\delta)}{12} \right] + (E^* - E_*)\, L_2(\delta) \left[x_2^2(\delta) + \frac{L_2^2(\delta)}{12} \right].$$

For the function $E(y)$ determined by (29), in choosing L_1, L_2 and x_1, x_2 according to (30), two conditions have been fulfilled: $I_0(E) = a$, $I_1(E) = b$. It only remains to fulfill the third condition of (27), $I_2(E) = c$, which in the case under consideration takes on the form of the equation with one unknown

$$I_2(\delta) = c\,. \tag{31}$$

Solving approximately Eq. (31) for δ with a given accuracy and using formulas (30), we obtain the design of a plate with given rigidities formed from two materials and containing two layers.

Let us exemplify the solution of the design problem. Let us use ten materials with Young moduli $(1, ..., 10)\cdot 10^{10}$ Pa and a Poisson coefficient $\nu = 0.3$. These values correspond to soft metals, ceramics, and rigid plastics. It is required to design a plate of thickness $h = 10^{-2}$ m and a longitudinal rigidity $S^0_{1111} = 4\cdot 10^8$, a zero asymmetric rigidity, and a flexural rigidity $S^0_{2222} = 1\cdot 10^4$. Estimates (28) permit choosing pairs of materials suitable for making a plate. In the given case, we have the following pairs of materials (the numbers of materials to which there correspond the values of the Young moduli $(1, ..., 10)\cdot 10^{10}$ Pa are given): (1, 9), (1, 10), (2, 9), (2, 10), (3, 9), (3, 10), (4, 9), (4, 10), (5, 10), and (6, 10). For each pair of materials, the simplest design of a plate with given rigidities has been obtained. Two of them are given in Tables 1 and 2 (the design presented on the last line consists of only three layers).

Physical Rigidities and Rigidities in an Arbitrary Coordinate System. Design of a Plate with Given Physical Rigidities. Calculating rigidities in an arbitrary coordinate system, we obtain longitudinal and flexural rigidities and, in the general case, nonzero asymmetric rigidities. In practice, a plate is characterized by only longitudinal and flexural rigidities. Let us denote them as D_0 and D_2 and call them physical rigidities. They differ by the factors depending on the Poisson coefficient (see formulas (1)). Here by D_0 and D_2 are meant rigidities with index 1111 in (1). Asymmetric rigidity is not a physical characteristic of the plate and is associated with an arbitrary choice of the coordinate system. Rigidities D_0 and D_2 are calculated in a coordinate system, in which symmetric rigidities are equal to zero. At a given structure of the plate, such a coordinate system can be found [7, 8] (for a laminated plate this is fairly easy to do). In the design problem, however, the structure of the plate is, in principle, not known in advance (and it is determined). Therefore, the coordinate system in which the symmetric rigidities are equal to zero is not known in the design problems.

The physical rigidities D_0 and D_2 are expressed in terms of the values of $I_0(E)$, $I_1(E)$, and $I_2(E)$ as follows [6]:

$$D_0 = h I_0(E)\,, \tag{32}$$

$$D_2 = h^3 \left[I_2(E) - \frac{I_1^2(E)}{I_0(E)} \right]. \tag{33}$$

The possible values of D_0 are obvious: $E_* h \le D_0 \le E^* h$. Let us find what values can the flexural rigidity assume D_2 (33). Transform expression (33):

$$\frac{D_2}{h^3} = I_2(E) - \frac{I_1^2(E)}{I_0(E)}. \tag{34}$$

Let us introduce, for brevity, the notations $I_0(E) = x$, $I_1(E) = y$, and $I_2(E) = z$. Taking them into account, the set of possible values of integrals $I_0(E)$, $I_1(E)$, and $I_2(E)$ (see [28]) is written as

$$E_* \le x \le E^*, \quad -j(x) \le y \le j(x), \quad I_*(x, y) \le z \le I^*(x, y), \tag{35}$$

where $j(x)$, $I_*(x, y)$, $I^*(x, y)$ are known functions (namely the functions from (27) and (28)). Then equality (34) will take the form

$$\frac{D_2}{h^3} = z - \frac{y^2}{x}.$$

Let us analyze the expression D_2/h^3 for the maximum and minimum on set (35). Note that for the plate with a zero asymmetric rigidity $y = 0$. It may be expected that the range of D_2/h^3 values on the set of functions (35) is wider than the set of values of the variable z. Let us show that in the given case this is not so. Fix the variable x, assuming $x = x_0$. Then

$$-j(x_0) \le y \le j(x_0), \quad I_*(x_0, y) \le z \le I^*(x_0, y), \quad \frac{D_2}{h^3} = z - \frac{y^2}{x_0}.$$

Find $\max \left(\frac{D_2}{h^3} \right)$. Taking into account that, at given (x_0, y), $\max x = I_*(x_0, y)$, we get

$$\max \frac{D_2}{h^3} = \max \left[I^*(x_0, y) - \frac{y^2}{x_0} \right].$$

Substituting the expression for $I^*(x_0, y)$ obtained in view of (27), (28) and the introduced change of variables, we get

$$\max \frac{D_2}{h^3} = \max \left\{ \left[\frac{E^*}{12} - \frac{(E^* - x_0)^3}{12(E^* - E_*)^2} \right] - y^2 \left[\frac{1}{E^* - x_0} + \frac{1}{x_0} \right] \right\}.$$

Since $\frac{1}{E^* - x_0} + \frac{1}{x_0} > 0$ (according to (35), $E^* - x_0 \ge 0$), the maximum value of D_2/h^3 is attained at $y = 0$. Let us find $\min \left(\frac{D_2}{h^3} \right)$ at fixed $x = x_0$ (at given (x_0, y) $\min z = I_*(x_0, y)$):

$$\min \frac{D_2}{h^3} = \min \left[I_*(x_0, y) - \frac{y^2}{x_0} \right].$$

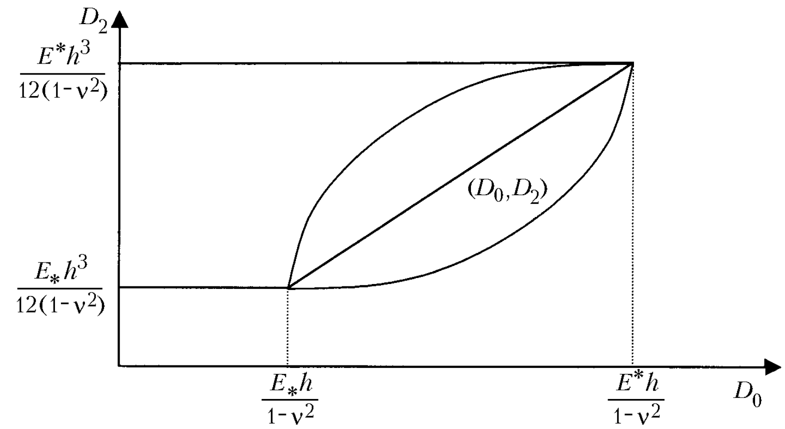

Fig. 3. Range of possible values of rigidities D_0, D_2.

Substitution of the expression from (28) for $I_*(x_0, y)$ yields

$$\min \frac{D_2}{h^3} = \min \left\{ \left[\frac{E_*}{12} + \frac{(x_0 - E_*)^3}{12\,(E^* - E_*)^2} \right] + y^2 \left[\frac{1}{x_0 - E_*} - \frac{1}{x_0} \right] \right\}.$$

Since $\frac{1}{x_0 - E_*} - \frac{1}{x_0} > 0$ (according to (35), $x_0 - E_* \geq 0$), the minimum value of D_2/h^3 is attained at $y = 0$.

As a result, we obtain that the minimum and maximum values of the expression D_2/h^3 are attained at $y = 0$, i.e., in the accepted designations this is the condition $I_1(E) = 0$ (the condition that the asymmetric rigidity of the plate is equal to zero). Thus, the possible values of the physical rigidities D_0 and D_2 coincide with the possible values of the integrals $I_0(E)$ and $I_2(E)$ at $b = 0$. Then, with account for the introduced designations, for the integrals $I_\mu(E)$, $\mu = 0, 1, 2$, we obtain that the physical rigidities D_0 and D_2 can take only the following values:

$$\frac{h}{1-\nu^2} E_* \leq D_0 \leq \frac{h}{1-\nu^2} E^*,$$

$$\frac{h^3}{1-\nu^2} \left[\frac{E_*}{12} + \frac{(a - E_*)^3}{12\,(E^* - E_*)^2} \right] \leq D_2 \leq \frac{h^3}{1-\nu^2} \left[\frac{E^*}{12} - \frac{(E^* - a)^3}{12\,(E^* - E_*)^2} \right]. \tag{36}$$

Formula (36) has been written for rigidities with indices 1111; for other indices one has to replace the factor $\frac{1}{1-\nu^2}$ by the corresponding quantity.

The value of the integral $I_1(E)$ does not influence the possible values of the physical rigidities D_0 and D_2. Then $I_1(E)$ (having the meaning of asymmetric rigidity) can be assumed to be equal to zero and the design problem can be formulated as

$$I_0(E) = D_0, \quad I_1(E) = 0, \quad I_2(E) = D_2 \tag{37}$$

without restricting the generality.

Problem (37) is solvable when conditions (36) are met and has the simplest ("four-step") solution. The range of possible values of a pair of rigidities (D_0, D_2) at a fixed thickness of the plate h is enclosed between two cubic parabolas with vortices at points $\left(\frac{E_* h}{1-\nu^2}, \frac{E_* h^3}{12\,(1-\nu^2)} \right)$ and $\left(\frac{E^* h}{1-\nu^2}, \frac{E^* h^3}{12\,(1-\nu^2)} \right)$ (Fig. 3). For a homogeneous plate, $D_2 = D_0 h/12$, which corresponds to the straight line in Fig. 3.

CONCLUSIONS

1. Using a finite set of materials, one can obtain a plate with any possible rigidity values.

2. The set of possible rigidity values is only determined by the maximum and minimum values of the Young moduli of materials E_*, E^*.

3. To make a plate with any possible set of rigidities, one can use only two materials distributed between four layers. To obtain a design, it suffices to solve the nonlinear algebraic equation.

4. In solving the design problem, one can assume the asymmetric rigidities to be equal to zero (i.e., exclude them from consideration), which will not limit the possible values of physical rigidities.

NOTATION

D, D_*^*, sets of functions; D_0, longitudinal rigidity; D_{ijkl}^0, rigidities in the plate plane; D_{ijkl}^1, asymmetric rigidities; D_2, D_{ijkl}^2, *flexural rigidities;* $E(y)$, Young modulus of the material as a function of the transverse coordinate y; E^* and E_*, maximum and minimum values of the Young modulus; h, thickness of the plate; $I_\mu(E)$, μth moment of the function $E(y)$; U_E, set of functions; δ, auxiliary variable; ν, Prandtl coefficient. Subscripts: i, j, k, l = 1, 2; m, medium group.

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